The Bootstrap Method

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Motivation

- **Normal Regression Model:** $y_i = \boldsymbol{x}_i'\beta + \varepsilon_i, \qquad \varepsilon_i \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$.
  - *MLE:* $\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$
  - *Confidence Intervals:* $\hat{\beta} \sim \mathcal{N}(\beta, \sigma^2\boldsymbol{V}), \quad \boldsymbol{V} = (\boldsymbol{X}'\boldsymbol{X})^{-1}$.

    $\Longrightarrow$ 95% CI for β_j is $\hat{\beta}_j \pm 1.96 \cdot \hat{\sigma} V_{jj}^{1/2}$, where $\hat{\sigma}$ is the MLE of σ.

    (This is the Observed Fisher Information method, which is indistinguishable from the exact CI based on the $t_{(n-p)}$ distribution for $n - p > 30$.)

Motivation

- **Normal Regression Model:** $y_i = \boldsymbol{x}_i'\beta + \varepsilon_i, \qquad \varepsilon_i \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$
- **Relaxed Assumptions:** $y_i = \boldsymbol{x}_i'\beta + \varepsilon_i, \qquad \varepsilon_i \overset{\text{iid}}{\sim} f(\varepsilon),$

$$E[\varepsilon_i] = 0, \ \text{var}(\varepsilon_i) = 1.$$

  - *Estimator:* Under Relaxed Assumptions, $\hat{\beta}$ is the Best Linear Unbiased Estimator (BLUE), in the sense that for any $\tilde{\beta} = \boldsymbol{A}\boldsymbol{y}$ with $E[\tilde{\beta}] = \beta$,

$$\text{var}(\boldsymbol{a}'\hat{\beta}) \le \text{var}(\boldsymbol{a}'\tilde{\beta}), \qquad \boldsymbol{a} \in \mathbb{R}^p.$$

  - *Confidence Intervals:* By linearity still have $\text{var}(\hat{\beta}) = \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$. Turns out that normality-based CI is asymptotically valid.

Motivation

- **Mean Regression:** $E[y \mid \boldsymbol{x}] = \boldsymbol{x}'\beta$.
- **Quantile Regression:** Define the τ-level quantile function

$$q_\tau(y \mid \boldsymbol{x}) = F^{-1}_{y|\boldsymbol{x}}(\tau \mid \boldsymbol{x}) \qquad \iff \qquad \Pr\{y \le q_\tau(y \mid \boldsymbol{x}) \mid \boldsymbol{x}\} = \tau.$$

The QR model is

$$q_\tau(y \mid \boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{\beta},$$

for any $\boldsymbol{x} \in \mathbb{R}^p$ and specific $\tau \in (0, 1)$ (or multiple τ each with their own $\boldsymbol{\beta}_\tau$).

Quantile Regression

Examples

1. **Additive Model:** $y = \boldsymbol{x}'\beta + \varepsilon$, where ε is an arbitrary error independent of $\boldsymbol{x}$.

   $\implies q_\tau(y \mid \boldsymbol{x}) = \boldsymbol{x}'\beta + q_\tau(\varepsilon)$.

Quantile Regression

Examples

2. **Location-Scale Model:** $y = \boldsymbol{x}'\boldsymbol{\gamma} + \boldsymbol{x}'\boldsymbol{\eta} \cdot \varepsilon, \quad \varepsilon \perp\!\!\!\perp \boldsymbol{x}$.

$$\implies q_\tau(y \mid \boldsymbol{x}) = \boldsymbol{x}'[\boldsymbol{\gamma} + \boldsymbol{\eta} \cdot q_\tau(\varepsilon)].$$

(Having $\boldsymbol{x}$ in both mean and standard deviation is not a real restriction, i.e., set $\boldsymbol{x} = (\boldsymbol{z}, \boldsymbol{w})$, $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_z, \mathbf{0})$, $\boldsymbol{\eta} = (\mathbf{0}, \boldsymbol{\eta}_w)$.)

Quantile Regression

Examples

3. **Fixed-Quantile Error:** $y = \boldsymbol{x}'\beta + \varepsilon$, where ε is not independent of $\boldsymbol{x}$, but

   $q_\tau(\varepsilon \mid \boldsymbol{x}) = \text{CONST}$.

   For example, $\varepsilon \mid \boldsymbol{x} \sim \sigma_{\boldsymbol{x}} \cdot t_{(\nu_{\boldsymbol{x}})}$, where $\nu_{\boldsymbol{x}}$ is arbitrary and

$$\sigma_{\boldsymbol{x}} = \frac{\text{CONST}}{\texttt{qt}(\tau, \texttt{df} = \nu_{\boldsymbol{x}})}$$

Quantile Regression
Examples

4. **Fully specified QR model:** $q_\tau(y \mid \boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{\beta}_\tau$ for all $0 < \tau < 1$.

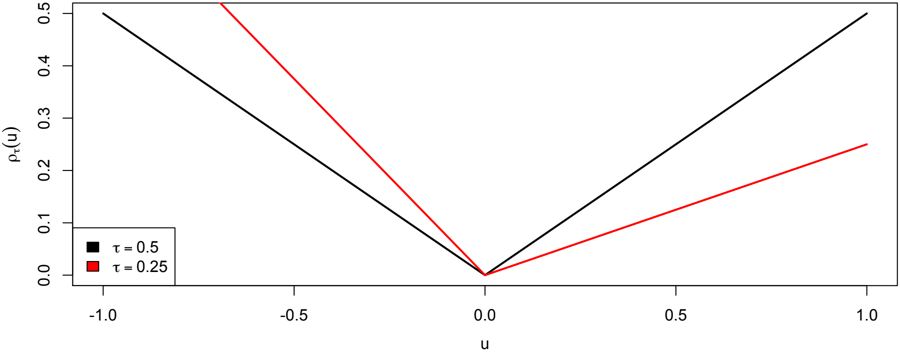
   Actually quite restrictive since quantiles need to be ordered:

$$\tau_1 < \tau_2 \qquad \Longrightarrow \qquad \boldsymbol{x}'\boldsymbol{\beta}_{\tau_1} < \boldsymbol{x}'\boldsymbol{\beta}_{\tau_2} \quad \forall\ \boldsymbol{x} \in \mathbb{R}^p.$$

Parameter Estimation

- **Quantile Regression Model:** $q_\tau(y \mid \boldsymbol{x}) = \boldsymbol{x}'\beta$ for given $\tau \in (0,1)$.
- **Moment Condition:** If true parameter value if $\beta = \beta_0$, then

$$\beta_0 = \arg\min_{\boldsymbol{\beta}} E[\rho_\tau(y - \boldsymbol{x}'\beta)], \qquad \rho_\tau(u) = u \cdot (\tau - \mathbb{1}\{u < 0\}).$$



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- **Sample Analog:**

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} \rho_\alpha(y_i - \boldsymbol{x}_i'\beta).$$

Parameter Estimation

- **Quantile Regression Model:** $q_\tau(\alpha \mid \boldsymbol{x}) = \boldsymbol{x}'\beta$ for given $\tau \in (0,1)$.
- **Point Estimate:**

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} \rho_\tau(y_i - \boldsymbol{x}_i'\boldsymbol{\beta}), \quad \rho_\tau(u) = u \cdot (\tau - \mathbb{1}\{u < 0\}).$$

Equivalent Formulation:

$$\min_{\boldsymbol{\beta}^+, \boldsymbol{\beta}^-, \boldsymbol{u}^+, \boldsymbol{u}^-} \sum_{i=1}^{n} \tau u_i^+ + (1-\tau) u_i^- \qquad \text{subject to} \qquad \boldsymbol{X}(\beta^+ - \beta^-) + \boldsymbol{u}^+ - \boldsymbol{u}^- = \boldsymbol{y},$$

where $\beta_j^+ = \max(\beta_j, 0)$, $\beta_j^- = -\min(\beta_j, 0)$ and similarly for u_i^+ and u_i^-.

Parameter Estimation

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$$\min_{\boldsymbol{\beta}^+,\boldsymbol{\beta}^-,\boldsymbol{u}^+,\boldsymbol{u}^-} \sum_{i=1}^n \tau u_i^+ + (1-\tau)u_i^- \qquad \text{subject to} \qquad \boldsymbol{X}(\beta^+ - \beta^-) + \boldsymbol{u}^+ - \boldsymbol{u}^- = \boldsymbol{y},$$

where $\beta_j^+ = \max(\beta_j, 0)$, $\beta_j^- = -\min(\beta_j, 0)$ and similarly for u_i^+ and u_i^-.

This is a **linear program** in $\boldsymbol{w} = (\boldsymbol{\beta}^+, \boldsymbol{\beta}^-, \boldsymbol{u}^+, \boldsymbol{u}^-)$,

$$\hat{\boldsymbol{w}} = \arg\min_{\boldsymbol{w}} \boldsymbol{c}'\boldsymbol{w} \qquad \text{subject to} \qquad \boldsymbol{A}\boldsymbol{w} \le \boldsymbol{b}, \boldsymbol{w} \ge \boldsymbol{0},$$

for which efficient algorithms are available.

Quantile Regression

- **Model:** $q_\tau(y \mid \boldsymbol{x}) = \boldsymbol{x}'\beta$ for given $\tau \in (0, 1)$.
- **Point Estimate:** $\hat{\beta} = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^n \rho_\tau(y_i - \boldsymbol{x}_i'\beta)$ via linear programming.
- **Confidence Intervals:** ???, since we don't have a likelihood to calculate Observed Fisher Information!
  - Add modeling assumptions $\implies \hat{\beta} \to \mathcal{N}(\beta_0, \boldsymbol{\Sigma})$, but $\boldsymbol{\Sigma}$ is difficult to estimate (nonparametric smoothing estimator with high variance).
  - Can do something much simpler... (but computationally more intensive)

The Problem

- **Data and Model:** $\boldsymbol{y} = (y_1, \dots, y_n) \sim F(\boldsymbol{y})$.

  I.e., completely general data-generating process (DGP) on the random vector $\boldsymbol{y}$. Could be a parametric model $\boldsymbol{y} \sim f(\boldsymbol{y} \mid \boldsymbol{\theta})$, a nonparametric model $y_i \overset{\text{iid}}{\sim} F(y)$, or a semi-parametric model like quantile regression...

- **Quantity of Interest:** $\tau_0 = \mathcal{G}(F)$.

  I.e., τ_0 must be some functional of the DGP. Could be $\tau_0 = \tau(\boldsymbol{\theta}_0)$, or the median of F...

- **Estimator:** $\hat{\tau} = g(\boldsymbol{y})$.

- **Objective:** Calculate a confidence interval for τ_0.

The Problem

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- **Quantity of Interest:** $\tau_0 = \mathcal{G}(F)$.
- **Estimator:** $\hat{\tau} = g(\boldsymbol{y})$.
- **Objective:** Calculate a confidence interval for τ_0.
- **Problem:** Can't use likelihood theory because:

1. Don't have a parametric likelihood $f(\boldsymbol{y} \mid \boldsymbol{\theta})$.
2. Have likelihood but estimator is not MLE (e.g., lasso for variable selection).
3. Have likelihood + MLE, but suspect some degree of model misspecification.
4. Have likelihood + MLE + correct model, but sample size n is too small for asymptotics to kick in.

The Bootstrap Method

- **Data and Model:** $\boldsymbol{y} = (y_1, \ldots, y_n) \sim F(\boldsymbol{y})$.
- **Quantity of Interest:** $\tau_0 = \mathcal{G}(F)$.
- **Estimator:** $\hat{\tau} = g(\boldsymbol{y})$.
- **Objective:** Calculate a confidence interval for τ_0.
- **Idealized Scenario:** Suppose an oracle gives you the distribution of the pivotal quantity $T = \tau_0 - \hat{\tau}$. Then

$$\Pr(L < T < U) = \Pr(L < \tau_0 - \hat{\tau} < U) = \Pr(\hat{\tau} + L < \theta_0 < \hat{\tau} + U).$$

$\implies$ If L/U are the 2.5/97.5% quantiles of T, then a 95% CI for τ_0 is

$$\tau_0 \in (\hat{\tau} + L, \hat{\tau} + U).$$

The Bootstrap Method

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- **Quantity of Interest:** $\tau_0 = \mathcal{G}(F)$.
- **Estimator:** $\hat{\tau} = g(\boldsymbol{y})$.
- **Oracle:** Suppose distribution of $T = \tau_0 - \hat{\tau}$ is given.

  If L/U are the 2.5/97.5% quantiles of T, then CI for τ_0 is $(\hat{\tau} + L, \hat{\tau} + U)$.
- **Bootstrap:** Estimate L and U as follows:
  1. **Simulate** M datasets $\tilde{\boldsymbol{y}}^{(m)} \overset{\text{iid}}{\sim} \hat{F}(\boldsymbol{y})$, each of size n, where $\hat{F}(\boldsymbol{y})$ is an estimate of $F(\boldsymbol{y})$. The two most common ways to do this are:
     - i. **Parametric Bootstrap:** If $\boldsymbol{y} \sim f(\boldsymbol{y} \mid \boldsymbol{\theta})$, then $\tilde{\boldsymbol{y}}^{(m)} \overset{\text{iid}}{\sim} f(\boldsymbol{y} \mid \hat{\boldsymbol{\theta}})$.
     - ii. **Nonparametric Bootstrap:** If $y_i \overset{\text{iid}}{\sim} F(y)$, then $y_i^{(m)} \overset{\text{iid}}{\sim} \hat{F}(y)$, where $\hat{F}(y)$ is the empirical CDF of $\boldsymbol{y}$. In other words, $\tilde{\boldsymbol{y}}^{(m)}$ is sampled n times with replacement from $\boldsymbol{y}$.

The Bootstrap Method

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The Bootstrap Method

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  2. For each dataset, calculate $\tilde{\tau}^{(m)} = g(\tilde{\boldsymbol{y}}^{(m)})$ and $\tilde{T}^{(m)} = \hat{\tau} - \tilde{\tau}^{(m)}$.
  3. Let $\tilde{L}/\tilde{U}$ be the 2.5/97% *sample* quantiles of $\tilde{T}^{(1)}, \ldots, \tilde{T}^{(M)}$.

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  $\Longrightarrow$ The Bootstrap CI for τ_0 is given by $(\hat{\tau} + \tilde{L}, \hat{\tau} + \tilde{U})$.

The Bootstrap Method

	Real World	Bootstrap World
Sampling Distribution	$\boldsymbol{y} \sim F(\boldsymbol{y})$	$\tilde{\boldsymbol{y}} \sim \hat{F}(\boldsymbol{y})$
Quantity of Interest	$\tau_0 = \mathcal{G}(F)$	$\hat{\tau} = g(\boldsymbol{y})$
Estimator	$\hat{\tau} = g(\boldsymbol{y})$	$\tilde{\tau} = g(\tilde{\boldsymbol{y}})$
Pivotal Quantity	$T = \tau_0 - \hat{\tau}$	$\tilde{T} = \hat{\tau} - \tilde{\tau}$
Quantiles:	$P(L < T < U) = 95\%$	$P(\tilde{L} < \tilde{T} < \tilde{U}) = 95\%$
95% Confidence Interval	Oracle: $(\hat{\tau} + L, \hat{\tau} + U)$ Bootstrap: $(\hat{\tau} + \tilde{L}, \hat{\tau} + \tilde{U})$	

Parallel between the Real world and the Bootstrap world.

Example: Range of Uniform

- **Objective:** Given $\boldsymbol{U} = (U_1, \ldots, U_n)$, $U_i \stackrel{\text{iid}}{\sim} \text{Unif}(0, \theta)$, we wish to estimate θ.
- **Simulation Study:** Generate $N = 1000$ datasets with $\theta_0 = 1$ and perform calculations for each of the following settings:

  2. *Estimators:* (i) MLE $\hat{\theta}_1 = \max(\boldsymbol{U})$ and (ii) Unbiased $\hat{\theta}_2 = 2\bar{\boldsymbol{U}}$ (since $E[\bar{\boldsymbol{U}}] = \theta/2$).

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  3. *Bootstrap Sampling:* (i) Nonparametric ($\tilde{\boldsymbol{U}}$ sampled with replacement) and (ii) Parametric ($\tilde{U}_i^{(m)} \overset{\text{iid}}{\sim} \text{Unif}(0, \hat{\theta})$). Always use $M = 1000$ bootstrap samples.

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  4. *Confidence Intervals:* (i) Basic Bootstrap: $(\hat{\theta} + \tilde{L}, \hat{\theta} + \tilde{U})$
  (ii) Percentile Bootstrap 2.5/97.5% quantiles of $\tilde{\theta}^{(1)}, \ldots \tilde{\theta}^{(M)}$. (seems simpler but...)

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  5. *Model Misspecification:* True sampling distribution is $U_i \overset{\text{iid}}{\sim} \theta \times \text{Beta}(\alpha, \alpha)$, where (i) $\alpha = 1$ ($\text{Beta}(1,1) = \text{Unif}(0,1)$) and (ii) $\alpha = 2$.
     (θ is range of distribution, so still meaningfull quantity to estimate. For $\alpha \neq 1$, $\hat{\theta}_1$ no longer MLE, but $\hat{\theta}_2$ still unbiased.)

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- **Comparison Metrics:** (i) True coverage of CI and (ii) Average width of CI.

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- **Objective:** Given $\boldsymbol{U} = (U_1, \ldots, U_n)$, $U_i \overset{\text{iid}}{\sim} \text{Unif}(0, \theta)$, wish to estimate θ.
- **Simulation Study:** For $N = 1000$ datasets with $\theta_0 = 1$:

  1. *Sample Size:* (i) $n = 100$ and (ii) $n = 10000$
  2. *Estimators:* (i) $\hat{\theta}_1 = \max(\boldsymbol{U})$ and (ii) $\hat{\theta}_2 = 2\bar{\boldsymbol{U}}$.
  3. *Bootstrap Sampling:* For $M = 1000$ bootstrap samples, sampling is

     i. Nonparametric: $\tilde{\boldsymbol{U}}$ sampled with replacement.

        Variance Reduction Use same $\tilde{\boldsymbol{U}}^{(m)}$ to calculate both $\hat{\theta}_1^{(m)}$ and $\hat{\theta}_2^{(m)}$.
        $\Longrightarrow$ Monte Carlo *difference* between comparison metrics has same expectation, but lower variance

     ii. Parametric: $\tilde{U}_i^{(m)} \overset{\text{iid}}{\sim} \text{Unif}(0, \hat{\theta})$.

  4. *Confidence Intervals:* (i) Basic Bootstrap and (ii) Percentile Bootstrap.
  5. *Model Misspecification:* $U_i \overset{\text{iid}}{\sim} \theta \times \text{Beta}(\alpha, \alpha)$, where (i) $\alpha = 1$ and (ii) $\alpha = 2$.

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     ii. Parametric: $\tilde{U}_i^{(m)} \overset{\text{iid}}{\sim} \text{Unif}(0, \hat{\theta})$.

        Variance Reduction Use same $\tilde{R}_i^{(m)} \overset{\text{iid}}{\sim} \text{Unif}(0, 1)$, and let $\tilde{U}_i^{(m)} = \hat{\theta}_k \tilde{R}_i^{(m)}$, $k = 1, 2$.

  4. *Confidence Intervals:* (i) Basic Bootstrap and (ii) Percentile Bootstrap.
  5. *Model Misspecification:* $U_i \overset{\text{iid}}{\sim} \theta \times \text{Beta}(\alpha, \alpha)$, where (i) $\alpha = 1$ and (ii) $\alpha = 2$.

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Example: Range of Uniform

Actual Coverage

alpha = 1

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.86	0.94	0.94	0.95
basic_n=10K	0.89	0.95	0.95	0.94
pct_n=100	0.00	0.95	0.00	0.95
pct_n=10K	0.00	0.95	0.00	0.95

alpha = 2

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.60	0.93	0.29	0.98
basic_n=10K	0.57	0.95	0.00	0.98
pct_n=100	0.00	0.94	0.00	0.98
pct_n=10K	0.00	0.95	0.00	0.99

Interval Width

alpha = 1

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.03	0.22	0.04	0.22
basic_n=10K	0.00	0.02	0.00	0.02
pct_n=100	0.03	0.22	0.04	0.22
pct_n=10K	0.00	0.02	0.00	0.02

alpha = 2

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.07	0.17	0.03	0.22
basic_n=10K	0.01	0.02	0.00	0.02
pct_n=100	0.07	0.17	0.03	0.22
pct_n=10K	0.01	0.02	0.00	0.02

Remarks:

1. Percentile CI based on $\hat{\theta}_1 = \max(\boldsymbol{U})$ has 0% coverage! This is because $\theta_0 > \hat{\theta}_1 > \tilde{\theta}_1^{(m)}$, so quantiles of $\tilde{\theta}_1^{(m)}$ can never cover θ_0.

Example: Range of Uniform

Actual Coverage

alpha = 1

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.86	0.94	0.94	0.95
basic_n=10K	0.89	0.95	0.95	0.94
pct_n=100	0.00	0.95	0.00	0.95
pct_n=10K	0.00	0.95	0.00	0.95

alpha = 2

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.60	0.93	0.29	0.98
basic_n=10K	0.57	0.95	0.00	0.98
pct_n=100	0.00	0.94	0.00	0.98
pct_n=10K	0.00	0.95	0.00	0.99

Interval Width

alpha = 1

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.03	0.22	0.04	0.22
basic_n=10K	0.00	0.02	0.00	0.02
pct_n=100	0.03	0.22	0.04	0.22
pct_n=10K	0.00	0.02	0.00	0.02

alpha = 2

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.07	0.17	0.03	0.22
basic_n=10K	0.01	0.02	0.00	0.02
pct_n=100	0.07	0.17	0.03	0.22
pct_n=10K	0.01	0.02	0.00	0.02

Remarks:

2. NP bootstrap with Basic CI does not approach 95% coverage as sample size $n \to \infty$! This is because bootstrap only works if $\tilde{\theta}$ and $\hat{\theta}$ have the same distribution as $n \to \infty$. However, $\hat{\theta}_1 \sim \theta_0 \times \text{Beta}(1, n)$ is a continuous distribution, but

$$\Pr(\tilde{\theta}_1 = \hat{\theta}_1) = 1 - \Pr(\tilde{\theta}_1 \neq \hat{\theta}_1) = 1 - (1 - \tfrac{1}{n})^n \to 1 - e^{-1} \approx 0.63.$$

Therefore, $\tilde{\theta}_1$ has a non-vanishing point mass at $\hat{\theta}_1$, so doesn't get close to continuous distribution of $\hat{\theta}_1$.

Example: Range of Uniform

Actual Coverage

alpha = 1

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.86	0.94	0.94	0.95
basic_n=10K	0.89	0.95	0.95	0.94
pct_n=100	0.00	0.95	0.00	0.95
pct_n=10K	0.00	0.95	0.00	0.95

alpha = 2

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.60	0.93	0.29	0.98
basic_n=10K	0.57	0.95	0.00	0.98
pct_n=100	0.00	0.94	0.00	0.98
pct_n=10K	0.00	0.95	0.00	0.99

Interval Width

alpha = 1

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.03	0.22	0.04	0.22
basic_n=10K	0.00	0.02	0.00	0.02
pct_n=100	0.03	0.22	0.04	0.22
pct_n=10K	0.00	0.02	0.00	0.02

alpha = 2

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.07	0.17	0.03	0.22
basic_n=10K	0.01	0.02	0.00	0.02
pct_n=100	0.07	0.17	0.03	0.22
pct_n=10K	0.01	0.02	0.00	0.02

Remarks:

3. NP-CI for $\hat{\theta}_2$ have the right coverage, even under wrong model $\alpha = 2$. On the other hand P-CI with $\hat{\theta}_2$ overcover under wrong model (98% instead of 95%).

Example: Range of Uniform

Actual Coverage

alpha = 1

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.86	0.94	0.94	0.95
basic_n=10K	0.89	0.95	0.95	0.94
pct_n=100	0.00	0.95	0.00	0.95
pct_n=10K	0.00	0.95	0.00	0.95

alpha = 2

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.60	0.93	0.29	0.98
basic_n=10K	0.57	0.95	0.00	0.98
pct_n=100	0.00	0.94	0.00	0.98
pct_n=10K	0.00	0.95	0.00	0.99

Interval Width

alpha = 1

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.03	0.22	0.04	0.22
basic_n=10K	0.00	0.02	0.00	0.02
pct_n=100	0.03	0.22	0.04	0.22
pct_n=10K	0.00	0.02	0.00	0.02

alpha = 2

	NP_max	NP_mean2	P_max	P_mean2
basic_n=100	0.07	0.17	0.03	0.22
basic_n=10K	0.01	0.02	0.00	0.02
pct_n=100	0.07	0.17	0.03	0.22
pct_n=10K	0.01	0.02	0.00	0.02

Remarks:

4. $\hat{\theta}_1$ does not converge to θ_0 under the wrong model $\alpha = 2$, so CI has poor coverage. On the other hand, interval width is narrower than with $\hat{\theta}_2$, because max has less variance than mean.

Example: GARCH Stochastic Volatility Model

- **SDE SV Model:** Let $(\Delta X_t, \Delta V_t)$ be the asset/volatility log-return/return on day t. The basic SDE-SV model is

$$\Delta X_t = (\alpha - \tfrac{1}{2} V_t)\Delta t + V_t^{1/2} \Delta B_{1t}$$
$$\Delta V_t = -\gamma(V_t - \mu)\Delta t + \sigma V_t^{1/2} \Delta B_{2t}$$

  - Pros: Excellent performance; easy to calibrate when V_t is observed (e.g., VIX for GSPC).
  - Cons: Extremely difficult to calibrate when V_t is latent, since $\ell(\boldsymbol{\theta} \mid \boldsymbol{X})$ is not available in closed-form, i.e,

$$\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{X}) \propto p(\boldsymbol{X} \mid \boldsymbol{\theta}) = \int p(\boldsymbol{X}, \boldsymbol{V} \mid \boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{V}$$

GARCH Stochastic Volatility Model

- **SDE SV Model:**

$$\Delta X_t = (\alpha - \tfrac{1}{2} V_t)\Delta t + V_t^{1/2} \Delta B_{1t}$$
$$\Delta V_t = -\gamma(V_t - \mu)\Delta t + \sigma V_t^{1/2} \Delta B_{2t}$$

- **GARCH SV Model:** Let $\varepsilon_t = \Delta X_t$. The GARCH(1,1) model is

$$\varepsilon_t = \sigma_t z_t, \qquad z_t \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$$
$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

  - Like SDEs, volatility σ_t is stochastic.
  - Pros: Inference with GARCH is far simpler than with SDE (closed-form likelihood).
  - Cons: Unlike SDEs, GARCH is a discrete-time model (difficult for option pricing and consistency across timescales)

GARCH Stochastic Volatility Model

- **Data:** Asset values $\boldsymbol{S} = (S_0, \ldots, S_N) \implies$ log-returns $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$, with $\varepsilon_t = \log(S_t/S_{t-1})$.
- **GARCH(1,1) Model:**

$$\varepsilon_t = \sigma_t z_t, \qquad z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$$
$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

- **Objective:** On given day N, estimate the p-day forward τ-level Value-At-Risk, i.e., the conditional quantile

$$\mathrm{VaR}_\tau = q_\tau\left(\frac{S_{N+p} - S_N}{S_N} \mid \boldsymbol{S}, \boldsymbol{\theta}\right) \quad \iff \quad \Pr\left(\frac{S_{N+p} - S_N}{S_N} < \mathrm{VaR}_\tau \mid \boldsymbol{S}, \boldsymbol{\theta}\right) = \tau.$$

For example, we would say that the 10-day 5%-level VaR of AAPL is a 1.3% drop in value.

GARCH Model

- **Model:** $\varepsilon_t = \sigma_t z_t, \qquad z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

- **Parameter Estimation:**
  - *R Packages* **rugarch**, **fGarch**. The former is more stable, the latter is faster. Both can fit numerous extensions to the basic GARCH(1,1) model above.
  - *Profile Likelihood* For $\boldsymbol{\theta} = (\omega, \alpha, \beta)$

$$\ell(\boldsymbol{\theta} \mid \boldsymbol{\varepsilon}) = -\frac{1}{2}\sum_{t=1}^{N} \frac{\varepsilon_t^2}{\sigma_t^2} + \log(\sigma_t^2), \qquad \sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

$$= -\frac{1}{2}\sum_{t=1}^{N} \frac{\varepsilon_t^2}{\omega \cdot \tilde{\sigma}_t^2} + \log(\omega \cdot \tilde{\sigma}_t^2), \qquad \tilde{\sigma}_t^2 = 1 + \eta\varepsilon_{t-1}^2 + \beta\tilde{\sigma}_{t-1}^2,$$

where $\eta = \alpha/\omega \implies \hat{\omega}(\eta, \beta) = \sum_{t=1}^{N}(\varepsilon_t/\tilde{\sigma}_t)^2$.

(Note the technical issue of initializing $\tilde{\sigma}_1$ which we won't discuss here.)

Value-at-Risk

- **GARCH(1,1) Model:** $$\varepsilon_t = \sigma_t z_t, \qquad z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$$ $$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$
- **Value-at-Risk:**

$$\mathsf{VaR}_\tau = q_\tau\left(\frac{S_{N+p} - S_N}{S_N} \mid \boldsymbol{S}, \boldsymbol{\theta}\right) \quad \iff \quad \Pr\left(\frac{S_{N+p} - S_N}{S_N} < \mathsf{VaR}_\tau \mid \boldsymbol{S}, \boldsymbol{\theta}\right) = \tau.$$

- **1-Day VaR:** For given $\boldsymbol{\theta}$ and data $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$

1. Let $\sigma_1^2 = E[\sigma_1^2 \mid \boldsymbol{\theta}] = \omega/(1 - \alpha - \beta)$

2. Use GARCH equation to obtain $\sigma_{N+1}^2 = \omega + \alpha\varepsilon_N^2 + \beta\sigma_N^2$

3. $(S_{N+1} - S_N)/S_N = \exp(\varepsilon_{N+1}) - 1 \implies \mathsf{VaR}_\tau = \exp\{\texttt{qnorm}(\tau \mid 0, \sigma_{N+1})\} - 1$

In other words, $\mathsf{VaR}_\tau = \mathsf{VaR}_\tau(\boldsymbol{\theta} \mid \boldsymbol{\varepsilon})$ is a function of $\boldsymbol{\theta}$ (and observed data $\boldsymbol{\varepsilon}$).

Value-at-Risk

- **GARCH(1,1) Model:** $\varepsilon_t = \sigma_t z_t, \qquad z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

- **1-Day VaR:** For given $\boldsymbol{\theta}$ and data $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$

  1. Let $\sigma_1^2 = E[\sigma_1^2 \mid \boldsymbol{\theta}] = \omega/(1-\alpha-\beta)$

  2. Use GARCH equation to obtain $\sigma_{N+1}^2 = \omega + \alpha\varepsilon_N^2 + \beta\sigma_N^2$

  3. $(S_{N+1} - S_N)/S_N = \exp(\varepsilon_{N+1}) - 1 \implies \text{VaR}_\tau = \exp\{\texttt{qnorm}(\tau \mid 0, \sigma_{N+1})\} - 1$

  In other words, $\text{VaR}_\tau = \text{VaR}_\tau(\boldsymbol{\theta} \mid \boldsymbol{\varepsilon})$.

- **Inference:** If $\hat{\boldsymbol{\theta}}$ is the MLE of GARCH model, then
  - *MLE:* Use plug-in principle: $\hat{\text{VaR}}_\tau = \text{VaR}_\tau(\hat{\boldsymbol{\theta}} \mid \boldsymbol{\varepsilon})$.
  - *Confidence Intervals?*

Delta-Method

- **Data-Generating Process:** Let $Y_1, Y_2, \ldots$ be some stochastic process determined by a parameter $\boldsymbol{\theta} \in \mathbb{R}^p$

  (In the simplest case, we have $Y_n \overset{\text{iid}}{\sim} f(y \mid \boldsymbol{\theta})$, but the theory works for stationary processes such as GARCH(1,1) as well).

- **Asymptotic Normality:** For $\boldsymbol{Y}_{1:n} = (Y_1, \ldots, Y_n)$, suppose the MLE and the inverse Fisher Information

$$\hat{\boldsymbol{\theta}}_n = \arg\max_{\boldsymbol{\theta}} \ell(\boldsymbol{\theta} \mid \boldsymbol{Y}_{1:n}), \qquad \hat{\boldsymbol{V}}_n = \left[-\tfrac{\partial^2}{\partial \boldsymbol{\theta}^2} \ell(\boldsymbol{\theta} \mid \boldsymbol{Y}_{1:n})\right]^{-1}$$

  satisfy the usual asymptotic theory, i.e., $\hat{\boldsymbol{V}}_n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_p)$ as $n \to \infty$, where $\boldsymbol{\theta}_0$ is the true parameter value.

Delta-Method

Theorem: Let $\hat{\boldsymbol{\theta}}_n$ be a sequence of estimators such that as $n \to \infty$ we have

$$\hat{\boldsymbol{V}}_n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}).$$

Suppose that $\boldsymbol{\tau} : \mathbb{R}^p \to \mathbb{R}^q$ is a continuously differentiable function with $q \le p$, and we wish to estimate $\boldsymbol{\tau}_0 = \boldsymbol{\tau}(\boldsymbol{\theta}_0)$. Then as $n \to \infty$ we have

$$\hat{\boldsymbol{\Sigma}}_n^{1/2}(\hat{\boldsymbol{\tau}}_n - \boldsymbol{\tau}_0) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}), \qquad \begin{aligned} \hat{\boldsymbol{\tau}}_n &= \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n) \\ \hat{\boldsymbol{\Sigma}}_n &= [\nabla \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)]' \hat{\boldsymbol{V}}_n [\nabla \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)]. \end{aligned}$$

Delta-Method

Theorem: Let $\hat{\boldsymbol{\theta}}_n$ be a sequence of estimators such that as $n \to \infty$ we have

$$\hat{\boldsymbol{V}}_n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}).$$

Suppose that $\boldsymbol{\tau} : \mathbb{R}^p \to \mathbb{R}^q$ is a continuously differentiable function with $q \leq p$, and we wish to estimate $\boldsymbol{\tau}_0 = \boldsymbol{\tau}(\boldsymbol{\theta}_0)$. Then as $n \to \infty$ we have

$$\hat{\boldsymbol{\Sigma}}_n^{1/2}(\hat{\boldsymbol{\tau}}_n - \boldsymbol{\tau}_0) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}), \qquad \hat{\boldsymbol{\tau}}_n = \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)$$
$$\hat{\boldsymbol{\Sigma}}_n = [\nabla \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)]' \hat{\boldsymbol{V}}_n [\nabla \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)].$$

Proof: The 1st order Taylor expansion of $\boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)$ about $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ gives

$$\boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n) - \boldsymbol{\tau}(\boldsymbol{\theta}_0) \approx [\nabla \boldsymbol{\tau}(\boldsymbol{\theta}_0)]'(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0).$$

Since $\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \approx \mathcal{N}(\boldsymbol{0}, \hat{\boldsymbol{V}}_n)$, by linearity of MVN we have

$$\hat{\boldsymbol{\tau}}_n - \boldsymbol{\tau}_0 \approx \mathcal{N}(\boldsymbol{0}, [\nabla \boldsymbol{\tau}(\boldsymbol{\theta}_0)]' \hat{\boldsymbol{V}}_n [\nabla \boldsymbol{\tau}(\boldsymbol{\theta}_0)]).$$

Delta-Method

- **Theorem:** If $\hat{\boldsymbol{V}}_n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ and $\boldsymbol{\tau} : \mathbb{R}^p \to \mathbb{R}^q$ is a continuously differentiable function with $q \le p$, then

$$\hat{\boldsymbol{\Sigma}}_n^{1/2}(\hat{\boldsymbol{\tau}}_n - \boldsymbol{\tau}_0) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}), \qquad \hat{\boldsymbol{\tau}}_n = \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)$$
$$\hat{\boldsymbol{\Sigma}}_n = [\nabla \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)]' \hat{\boldsymbol{V}}_n [\nabla \boldsymbol{\tau}(\hat{\boldsymbol{\theta}}_n)].$$

- **Upshot:** If $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{V}})$ are the MLE and its variance estimator, a confidence interval for a 1D quantity of interest $\tau_0 = \tau(\boldsymbol{\theta}_0)$ can be constructed via

$$\hat{\tau} \pm 1.96 \cdot s_{\hat{\tau}}, \qquad \hat{\tau} = \tau(\hat{\boldsymbol{\theta}})$$
$$s_{\hat{\tau}} = \sqrt{[\nabla \tau(\hat{\boldsymbol{\theta}})]' \hat{\boldsymbol{V}} [\nabla \tau(\hat{\boldsymbol{\theta}})]}.$$

  Can use this to calculate CI for 1-day $\mathrm{VaR}_\tau = \tau(\boldsymbol{\theta}_0) = \mathrm{VaR}_\tau(\boldsymbol{\theta}_0 \mid \boldsymbol{\varepsilon})$.

Value-at-Risk

- **GARCH(1,1) Model:**
$$\varepsilon_t = \sigma_t z_t, \qquad z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$$
$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$
- **Value-at-Risk:**
$$\mathrm{VaR}_\tau = q_\tau\left(\frac{S_{N+p}-S_N}{S_N} \mid \boldsymbol{S}, \boldsymbol{\theta}\right) \quad\iff\quad \Pr\left(\frac{S_{N+p}-S_N}{S_N} < \mathrm{VaR}_\tau \mid \boldsymbol{S}, \boldsymbol{\theta}\right) = \tau.$$
  No analytic solution for $p > 1$.
- **Point Estimate:** Use Monte Carlo:
  1. For given $\boldsymbol{\theta}$, analytically obtain $\sigma_1^2, \ldots, \sigma_N^2$.
  2. Generate M iid realizations of $R = \log(S_{N+p}/S_N)$ from $p(R \mid \varepsilon_N, \sigma_N)$ using GARCH. (Note that $R = \sum_{i=1}^p \varepsilon_{N+1}$)
  3. The Monte Carlo approximation is $\mathrm{VaR}_\tau = \exp\{\hat{q}_\tau(R \mid \varepsilon_N, \boldsymbol{\theta})\} - 1$, where $\hat{q}_\tau(R \mid \varepsilon_N, \boldsymbol{\theta})$ is the τ-level sample quantile of the iid realizations $R^{(1)}, \ldots, R^{(M)}$.
- **Interval Estimate:** Use Delta-Method, with $\hat{\mathrm{VaR}}_\tau = \exp\{\hat{q}_\tau(R \mid \varepsilon_N, \hat{\boldsymbol{\theta}})\} - 1$, but with variance reduction, i.e., same $z_{N+1}^{(m)}, \ldots, z_{N+p}^{(m)}$ for every value of $\boldsymbol{\theta}$.

Value-at-Risk

- **GARCH(1,1) Model:** $$\varepsilon_t = \sigma_t z_t, \qquad z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$$ $$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$
- **Value-at-Risk:**
$$\text{VaR}_\tau = q_\tau\left(\frac{S_{N+p} - S_N}{S_N} \mid \boldsymbol{S}, \boldsymbol{\theta}\right) \iff \Pr\left(\frac{S_{N+p} - S_N}{S_N} < \text{VaR}_\tau \mid \boldsymbol{S}, \boldsymbol{\theta}\right) = \tau.$$
- **Point/Interval Estimate:** Monte Carlo + Delta Method
- **Model Misspecification:** Suppose we have GARCH(1,1), but with $z_t \overset{\text{iid}}{\sim} F(z)$ with $F \neq \mathcal{N}(0,1)$?
- **Residual Bootstrap:**
  - GARCH model: $\varepsilon_t = \sigma_t z_t, \qquad \sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$
  - Use $\hat{\boldsymbol{\theta}}$ to calculate $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_N)$ and *residuals* $\hat{\boldsymbol{z}} = (\hat{z}_1, \ldots, \hat{z}_N) = \boldsymbol{\varepsilon}/\hat{\boldsymbol{\sigma}}$.
  - Obtain Bootstrap residuals $\tilde{\boldsymbol{z}}$ by sampling with replacement from $\hat{\boldsymbol{z}}$
  - Bootstrap log-returns: $\tilde{\varepsilon}_t = \tilde{\sigma}_t\tilde{z}_t, \qquad \sigma_t^2 = \hat{\omega} + \hat{\alpha} + \tilde{\varepsilon}_{t-1}^2 + \hat{\beta}\tilde{\sigma}_{t-1}^2$